Generalized relation between pulsed and continuous measurements in the quantum Zeno effect

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A relation is found between pulsed measurements of the excited state probability of a two-level atom illuminated by a driving laser, and a continuous measurement by a second laser coupling the excited state to a third state which decays rapidly and irreversibly. We find the time between pulses to achieve the same average detection time than a given continuous measurement in strong, weak, or intermediate coupling regimes, generalizing the results in L. S. Schulman, Phys. Rev. A **57**, 1509 (1998).

I. INTRODUCTION

The Quantum Zeno effect has not ceased to attract attention since its discovery [1, 2] due to its importance as a fundamental phenomenon and more recently because of different applications, in particular in quantum information [3, 4, 5] or thermodynamic control [6, 7]. Basic properties of the effect are being scrutinized both theoretically and experimentally, see for recent work [8, 9, 10, 11, 12] and references therein. One of the relevant and recurrent issues is the role played by the repeated measurements (questioning their necessity [13]) and the possibility to achieve similar results with continuous measurements. The first papers on the quantum Zeno effect formulated it as the suppression of transitions between quantum states because of continuous observation, *mimicked by* repeated instantaneous measurements [1, 2]. This suppression was taken as an argument against such a discretized description of the continuous measurement by Misra and Sudarshan [2] but, in later works, many authors have emphasized instead the connections and similarities between pulsed and continuous models, see [14, 15, 16, 17, 18] and references therein, in which the transition is affected by the measurement in various degrees depending on the parameters characterizing the system and the coupling with the measurement device included in the Hamiltonian. Schulman found an elegant relation between discrete and continuous models. In the simplest case the continuous model represents a laser-driven two-level atom with internal states $|1\rangle, |2\rangle$ subjected to an irreversible measurement of state $|2\rangle$ [17] with an intense (strong) coupling or quenching laser, see Fig. 1, or, equivalently, a fast natural decay of level 2 which, possibly counterintuitively, hinder the 1-2 transition in spite of the 1-2 driving laser. (In this work the word "coupling" is exclusively associated with the measurement, 2-3 transition, whereas the "driving"

refers to the 1-2 transition.) In the corresponding pulsed measurement to be compared with the continuous one, the 1-2 driving remains but, instead of the continuous measurement via effective or natural decay, an instantaneous measurement of level 2 is made every δt. Schulman noticed that if δt is chosen as four times the inverse of the effective decay rate, similar results are obtained in the continuous and pulsed cases, as it has later been observed experimentally [9]. We shall extend this result by providing a generalized connection between the continuous and the pulsed measurements which is also valid in the weak coupling regime, in approximate (explicit) or exact (implicit) forms.

II. THE MODEL

A. Continuous measurement

For the continuous model we assume a driving laser of frequency ω_{L12} for the 1-2 transition (with transition frequency ω_{12}, and Rabi frequency ω), and a coupling laser of frequency ω_{L23} for the 2-3 transition (with transition frequency ω_{23} and Rabi frequency Ω). The atom in 3 will emit a photon decaying irreversibly outside the original 1-2 subspace. In a recent experiment this irreversible decay has been realized by the recoil imparted at the photon emission, which gives the atom enough kinetic energy to escape from the confining trap [9]. We shall adopt a number of standard approximations for the initial Hamiltonian: semiclassical description for the two lasers, dipole approximation, and rotating-wave approximation,

$$
\begin{aligned}
H = {} & \hbar\omega_{12}|2\rangle\langle 2| + \hbar(\omega_{12} + \omega_{23} - i\Gamma/2)|3\rangle\langle 3| \\
& + \frac{\hbar}{2}\omega[|2\rangle\langle 1|e^{-i\omega_{L12}t} + H.c.] \\
& + \frac{\hbar}{2}\Omega[|3\rangle\langle 2|e^{-i\omega_{L23}t} + H.c.],
\end{aligned}
\tag{1}
$$

where $H.c.$ means Hermitian conjugate, and Γ is the decay rate of level 3. In a laser-adapted interaction picture, using $H_0 = |2\rangle\langle 2|\hbar\omega_{L12} + |3\rangle\langle 3|\hbar(\omega_{L12} + \omega_{L23})$, we
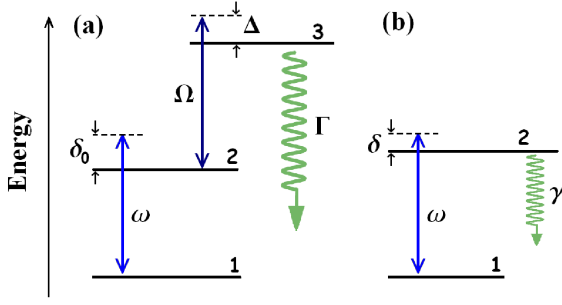
FIG. 1: (Color online) (a) Energy levels 1, 2, 3; detunings of the two lasers with respect to the 1-2 and 2-3 transitions; on-resonance Rabi frequencies, and decay rate of level 3; (b) Effective energy scheme after the adiabatic elimination of level 3. Note the level shift with respect to the scheme in (a).

may eventually write down a time-independent effective Hamiltonian,

$$H_I = \frac{\hbar}{2} \begin{pmatrix} 0 & \omega & 0 \\ \omega & -2\delta_0 & \Omega \\ 0 & \Omega & -i\Gamma - 2(\Delta + \delta_0) \end{pmatrix}. \quad (2)$$

Here δ_0 is the detuning (laser frequency minus transition frequency) for the 1-2 transition, and Δ is the detuning for the 2-3 transition. For a large enough $\Gamma \gg \Omega$, level 3 is scarcely populated (low saturation) and it can be adiabatically eliminated to obtain an effective Hamiltonian for the subspace of levels 1-2,

$$H_I = \frac{\hbar}{2} \begin{pmatrix} 0 & \omega \\ \omega & -i\gamma - 2\delta \end{pmatrix}, \quad (3)$$

where the effective decay rate γ is related to the measurement coupling parameters [9],

$$\gamma = \frac{\Gamma\Omega^2}{\Gamma^2 + 4\tilde{\Delta}^2}, \quad (4)$$

and

$$\delta = \delta_0 - \tilde{\Delta}\Omega^2/(4\tilde{\Delta}^2 + \Gamma^2), \quad (5)$$

with $\tilde{\Delta} = \Delta + \delta_0$. To obtain Eqs. (4, 5) using complex potentials see [19]. The general solution of the time-dependent Schrödinger equation corresponding to the Hamiltonian (3) is easy to find by Laplace transform. For the boundary condition $\psi^{(1)}(t = 0) = 1$, corresponding to the atom being initially in the ground state, the ground and excited state amplitudes take the form

$$\psi^{(1)} = e^{-\gamma_0 t/4} \left[\cosh(tR/2) + \frac{\gamma_0}{2R} \sinh(tR/2) \right], \quad (6)$$

and

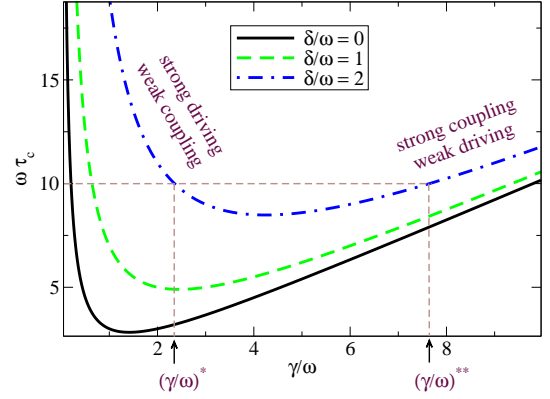$$\psi^{(2)} = -i\frac{\omega e^{-\gamma_0 t/4}}{R} \sinh(tR/2), \quad (7)$$

FIG. 2: (Color online) Dependence of average lifetime τ_c of a two-level atom continuously driving by a laser (of Rabi frequency ω) on the decay rate γ for different values of the detuning δ. A given value of τ_c can be generally achieved by applying weak, $(\gamma/\omega)^*$, or strong coupling, $(\gamma/\omega)^{**}$.

where

$$R = (\gamma_0^2 - 4\omega^2)^{1/2}/2, \quad (8)$$

$$\gamma_0 = -2i\delta + \gamma, \quad (9)$$

and the corresponding probabilities p_j are given as $|\psi^{(j)}|^2$, $j = 1, 2$.

The detection probability per unit time in the continuous measurement is $W(t) = \gamma|\psi^{(2)}(t)|^2$, and the average lifetime before detection, τ_c, is given by $\int_0^\infty dt\, tW(t)$. Using Eq. (7),

$$\tau_c = \frac{2}{\gamma} + \frac{\gamma}{\omega^2} + \frac{4\delta^2}{\gamma\omega^2}. \quad (10)$$

As shown in Fig. 2, the upshot is that the same average lifetime τ_c can be achieved for two different ratios of γ/ω, this is, both in the strong and weak driving regime. While in the former the system undergoes damped Rabi Oscillations (Fig. 3a), in the later the total probability approximately coincides with that in the ground state $p_{tot}(t) \cong p_1$ (Fig. 3b). Nonetheless, in both regimes the total probability obeys essentially the same decay law (Fig. 3c).

B. Pulsed measurement

Alternatively, a pulsed measurement of the population at 2 may be performed on the same two-level atom described before, which evolves between measurements with Hamiltonian

$$H_I(\Omega = 0) = \frac{\hbar}{2} \begin{pmatrix} 0 & \omega \\ \omega & -2\delta_0 \end{pmatrix}, \quad (11)$$

equal to (3) when $\Omega = 0$. Note the level shift with respect to the Hamiltonian (3) because of the absence of the 2-3 coupling.
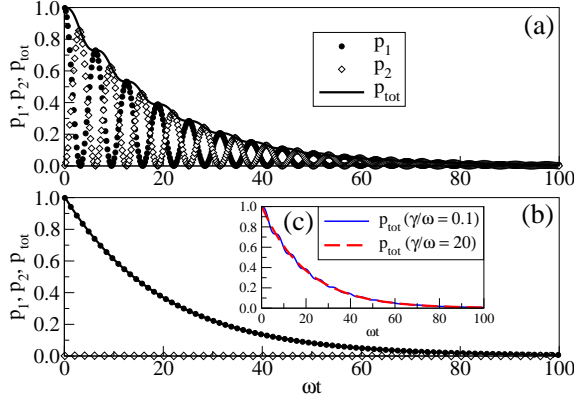
FIG. 3: (Color online) Evolution of the populations in an effective two-level atom driven by a laser with $\delta = 0$, in the (a) weak coupling ($\gamma/\omega=0.1$) and (b) strong coupling ($\gamma/\omega = 20$) regimes. Both cases lead to the same lifetime τ_c and the total populations $p_{tot} = p_1 + p_2$ are very similar, see (c).

At intervals δt, the population in 2 is measured "instantly" (in practice, in a time scale τ_m small compared to δt, but we shall ignore here the corrections of order τ_m/δ_t, see e.g. [9].) and removed from the atomic ensemble. This process is represented by successive projections onto the state $|1\rangle$ at times $\delta t, 2\delta t, ..., k\delta t, ...$ The
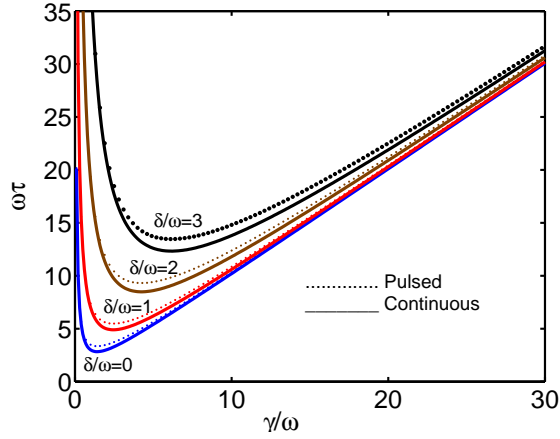

FIG. 4: (Color online) (a) Lifetimes before detection for continuous ($\tau = \tau_c$, solid lines) and pulsed ($\tau = \langle t\rangle$ using Eqs. (13) and (15), dashed lines) measurements for different values of detuning. For the pulsed calculations we have taken $\delta_0 = \delta$ corresponding to $\tilde{\Delta} = 0$.

probability to detect the atom in the k-th measurement instant is the probability to find the atom in level 2 at that instant, $P_2(1 - P_2)^{k-1}$, where we have introduced P_2, to be distinguished from the more generic p_2, as the probability to find the excited state at δt without coupling,

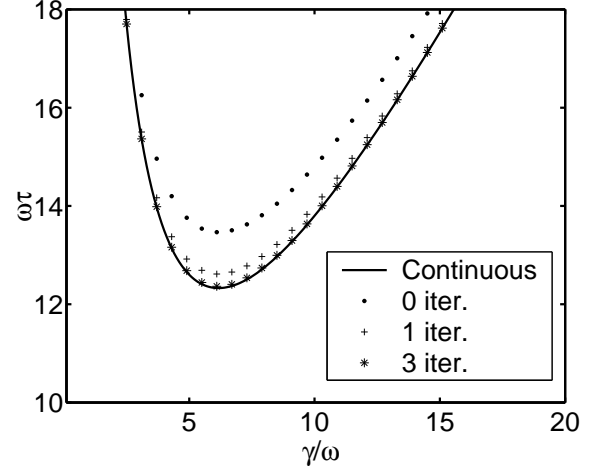$$P_2 \equiv |\psi^{(2)}(\Omega = 0, t = \delta t)|^2, \tag{12}$$

FIG. 5: Lifetimes before detection for $\delta/\omega = 3$. Solid and dotted lines are, as in Fig. 4, for the continuous measurement and for pulsed measurement, using (15), respectively. The symbols represent the first and third iteration of Eq. (18), starting with the seed of Eq. (15).

when the (normalized) state is in the ground state at $t = 0$. The average detection time is thus calculated, exactly, as

$$\langle t \rangle = \sum_{k=0}^{\infty} k\delta t P_2(1 - P_2)^{k-1} = \frac{\delta t P_2}{1 - P_2} \sum_{k=0}^{\infty} k(1 - P_2)^k$$
$$= \delta t/P_2, \tag{13}$$

where the last sum can be performed by taking the derivative of the geometric series.

C. Connection between continuous and pulsed measurements

The objective here is to find the time between pulses, δt, so that the average detection time in the pulsed measurement, $\langle t\rangle$, equals the lifetime τ_c of a given continuous measurement. Let us first pay attention to the short-time asymptotics for $H_I(\Omega = 0)$. From Eq. (7), with $\Omega = 0$, the dominant term for the probability of level 2 at δt, P_2, is independent of δ_0 (a remarkable result that only holds at short times), and takes the simple form

$$P_2 \sim \delta t^2 \omega^2/4 = (\delta t/\tau_Z)^2. \tag{14}$$

The short-time dependence is characterized by a "Zeno time" $\tau_Z \equiv 2/\omega$ which defines the time-scale for the quadratic time dependence [17]. Equating τ_c and $\langle t\rangle$, see Eqs. (10) and (13), with the quadratic approximation (14) for P_2 we get

$$\delta t = \frac{4\gamma}{2\omega^2 + \gamma^2 + 4\delta^2}. \tag{15}$$

The same result may be obtained "à la Schulman" [17] as follows: after k successive pulses separated by $\delta t << \tau_Z$, at $t = k\delta t$, the probability of the ground state (which is the total remaining probability after removing p_2 in each pulse) will assume the form

$$p_1(t) \approx \left(1 - \frac{t}{k}\frac{\delta t}{\tau_Z^2}\right)^k \approx \exp(-t/\tau_{EP}), \qquad (16)$$

i.e., an effective exponential decay law $\exp(-t/\tau_{EP})$ where $\tau_{EP} = \tau_Z^2/\delta t$ is the effective lifetime. By equating τ_{EP} and τ_c we get again Eq. (15). This requires $\delta t << 2/\omega$ or $\tau_Z << \tau_c$. These inequalities and therefore the exponential approximation are well fulfilled for weak 1-2 driving, $\omega/\gamma << 1$ (corresponding to strong 2-3 coupling), or strong 1-2 driving conditions $\omega/\gamma >> 1$ (corresponding to weak 2-3 coupling), but not so well for intermediate driving conditions, see the region near the minimum at $\gamma/\omega = [2+4(\delta/\omega)^2]^{1/2}$ in Fig. 4. For a more accurate treatment we may find the really optimal δt by imposing the equality between the exact expressions for the lifetimes, $\tau_c = \langle t \rangle$, and solving the implicit equation (13), e.g. by iteration,

$$\delta t = \tau_c P_2(\delta t). \qquad (17)$$

This may be carried out by a Newton-Raphson method,

$$\delta t = \frac{P_2(\delta t_0) - \delta t_0 P_2'(\delta t_0)}{\tau_c^{-1} - P_2'(\delta t_0)}, \qquad (18)$$

where the δt calculated on the left hand side becomes the new δt_0 for the following iteration and the prime means derivative with respect to δt. A good seed for the first δt_0 may be Eq. (15), and P_2' is very well approximated by $\delta t_0 \omega^2/2$. Applying Eq. (18) just once in this way provides a very good, approximate, but explicit expression of δt, and the corresponding $\langle t \rangle$, Eq. (13), is quite close to the continuous measurement result τ_c. Fig. 5 is a closeup of one of the cases in Fig. 4 and depicts a few iterations. Three of them are enough to produce a $\langle t \rangle$ indistinguishable from τ_c in the scale of the figure.

III. DISCUSSION

From Eq. (10) we see that the average detection time in the continuous measurement may be large both for strong measurement coupling and in the opposite weak-coupling case. A given lifetime in a Zeno-type pulsed measurement may thus be reproduced with continuous measurement by means of different conditions. For a fixed detuning δ and Rabi frequency, these are generally realized with two different values of γ for which the sum $p_1 + p_2$ (probability for the atom to remain undetected) is quite similar. This may have practical consequences as a control mechanism, since a strong continuous coupling could be difficult to produce or cause some undesirable effects. For example, if we want to maintain the effective
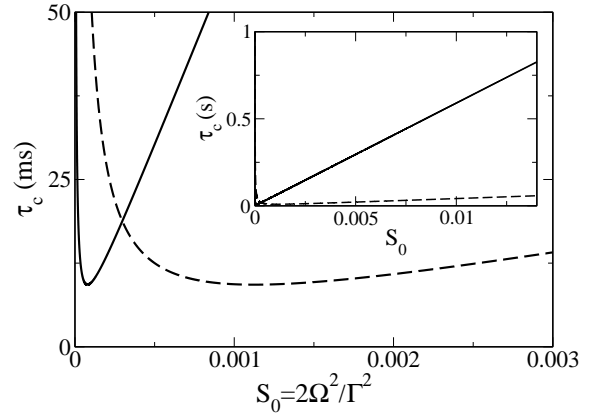

FIG. 6: Lifetimes before detection versus the saturation parameter $s_0 = 2\Omega^2/\Gamma^2$. $\omega = 2\pi \times 48.5$ Hz, $\Gamma = 2\pi \times 1.74$ MHz. Solid line: $\Delta = \delta_0 = \delta = 0$; Dashed line: $\Delta = 2\pi \times 3.18$ MHz, and δ_0 is adjusted for each s_0 so that $\delta = 0$.

two-level scenario, Ω cannot be made arbitrarily large in the scale of Γ to avoid a significant population of level 3 which would make the adiabatic elimination invalid. One more problem could be the excitation of other levels, breaking down, again, the simple two (or even three) level picture.

We have in summary generalized Schulman's relation between pulsed and continuous measurement in a two-level system in different ways: Schulman's result corresponds to the case $\delta = 0$, with $\gamma/\omega >> 1$, for which $\delta t = 4/\gamma$. Our relation (15) is not restricted to these conditions and is generally applicable. It is, however, an approximate one, since it relies on an approximate expression for the average measurement time in the pulsed case. We have also provided the exact result, by solving the implicit equation (17). The present results should not be difficult to verify experimentally with very similar settings as the ones already used in [9]. Fig. 6 shows the average continuous time τ_c versus the saturation $s_0 = 2\Omega^2/\Gamma^2$ using parameters from [9], see the caption for details. The solid line is for the case in which the detunings of the two transitions are set to zero, namely $\delta = \Delta = \delta_0 = 0$. In the dashed line the measuring laser detuning is fixed to $\Delta = -20 \times 10^6$ s^{-1}, and δ_0 is adjusted for each measuring laser intensity (i.e., for each value of s_0) so that $\delta = 0$. In principle this requires to solve a qubic equation for δ_0, see Eq. (5), but an excellent approximation for the parameter values in [9] (low saturation) is obtained by taking $\tilde{\Delta} \approx \Delta$,

$$\delta_0(\delta = 0) \approx \frac{\Delta \Omega^2}{4\Delta^2 + \Gamma^2}. \qquad (19)$$

In practice $\delta_0(\delta = 0)$ may be found by minimizing τ_c (or maximizing the reduction of the population of $|1\rangle$ [9]) with respect to the rf frequency ω_{12}. We plot this case to show that the weak driving region is accessible with parameters very near the ones used in [9]. The devia-

tion from the straight line is clearly noticeable around and below $s_0 = .001$ which corresponds in [9] to an optical power of 0.83 μW. In all cases we have solved the dynamics of the three-level system numerically and of the effective two-level system analytically without finding any disagreement between the two models in the scale of the figure.

Acknowledgments

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